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Test Spaces and Characterizations of Quadratic Spaces

Anatolij Dvurečenskij[1]

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We show that a test space consisting of nonzero vectors of a quadratic space E and of the set all maximal orthogonal systems in E is algebraic iff E is Dacey or, equivalently, iff E is orthomodular. In addition, we present another orthomodularity criteria of quadratic spaces, and using the result of Solèr, we show that they can imply that E is a real, complex, or quaternionic Hilbert space.

1. INTRODUCTION

Foulis and Randall (1972) presented the mathematical foundations of operational probability theory and statistics based upon a generalization of the conventional notion of a sample space. They generalized the approach of Kolmogorov (1933).

Let X be a nonvoid set; elements of X are called *outcomes*. We say that a pair $(X, \mathcal{T})$ is a *test space* iff $\mathcal{T}$ is a nonempty family of subsets of X such that (i) for any $x \in X$, there is a $T \in \mathcal{T}$ containing x, and (ii) if $S, T \in \mathcal{T}$ and $S \subseteq T$, then $S = T$.

Any element of $\mathcal{T}$ is said to be a *test*. We say that a subset G of X is an *event* iff there is a test $T \in \mathcal{T}$ such that $G \subseteq T$. Let us denote the set of all effects in X by $\mathcal{E} = \mathcal{E}(X, \mathcal{T})$. We say that two events F and G are (i) *orthogonal* to each other, in symbols $F \perp G$, iff $F \cap G = \emptyset$, and there is a test $T \in \mathcal{T}$ such that $F \cup G \subseteq T$, (ii) *local complements* of each other, in symbols F *loc* G, iff $F \perp G$ and there is a test $T \in \mathcal{T}$ such that $F \cup G = T$, (iii) *perspective with axis* H iff they share a common local complement H. We write $F \approx_H G$ or $F \approx G$ if the axis is not emphasized.

[1] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, SK-814 73 Bratislava, Slovakia; e-mail: dvurecen@mau.savba.sk.

The test space $(X, \mathcal{T})$ is *algebraic* iff, for $F, G, H \in \mathscr{E}$, $F \approx G$ and F *loc* H entail G *loc* H.

For algebraic test spaces, $\approx$ is the relation of an equivalence and, for any $A \in \mathscr{E}(X, \mathcal{T})$, we put

$$\pi(A) := \{B \in \mathscr{E}(X, \mathcal{T}): B \approx A\} \tag{1.1}$$

Then the logic of the algebraic test space $\mathscr{E}(X, \mathcal{T})$, i.e., the set

$$\Pi(X) := \{\pi(A): A \in \mathscr{E}(X, \mathcal{T})\} \tag{1.2}$$

is an orthoalgebra (Gudder, 1988; Foulis and Bennett, 1993). We recall that an *orthoalgebra* is a set L with two particular elements 0, 1, and with a partial binary operation $\oplus$: $L \times L \to L$ such that, for all $a, b, c \in L$, we have:

(OAi) If $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$ (*commutativity*).

(OAii) If $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (*associativity*).

(OAiii) For any $a \in L$, there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (*orthocomplementation*).

(OAiv) If $a \oplus a$ is defined, then $a = 0$ (*consistency*).

Let a and b be two elements of an orthoalgebra L. We say that (i) a is *orthogonal* to b and write $a \perp b$ iff $a \oplus b$ is defined in L, (ii) a is *less than or equal* b and write $a \le b$ iff there exists an element $c \in L$ such that $a \perp c$ and $a \oplus c = b$ (in this case, we also write $b \ge a$), (iii) b is the *orthocomplement* of a iff b is a (unique) element of L such that $b \perp a$ and $a \oplus b = 1$ and it is written as $a^\perp$.

We recall that an *orthomodular poset* (OMP for short) is an orthoalgebra L such that $a \perp b$ for $a, b \in L$ implies $a \vee b \in L$; if this is a case, then $a \vee b = a \oplus b$; if an OMP is a lattice, we call it an *orthomodular lattice* (OML for short).

We say that two orthoalgebras L_1 and L_2 are *isomorphic* iff there is a bijective mapping ϕ: $L_1 \to L_2$ such that $\phi(1) = 1$, and $\phi(a) \perp \phi(b)$ iff $a \perp b$, and then $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

If L is an orthoalgebra, put $X = L\setminus\{0\}$ and let $\mathcal{T}$ consist of all finite decompositions of 1, i.e., of all finite systems $\{a_i\}$ in X such that $\oplus_i a_i = 1$. Then $(X, \mathcal{T})$ is an algebraic test space, and $\Pi(X)$ is isomorphic with L (Gudder, 1988; Foulis and Bennett, 1993).

For mathematical foundations of quantum mechanics, the system of all closed subspaces $L(H)$ of a real, complex, or quaternionic Hilbert space H plays an important role. It is well known that $L(H)$ is a complete orthomodular, atomistic, irreducible lattice with covering property. If X is the unit sphere in H and $\mathcal{T}$ is the system of all orthonormal bases in H, then this test space is algebraic, and $\Pi(X)$ is isomorphic to $L(H)$. We see that test spaces generalize both Kolmogorov axiomatic as well as Hilbert-space quantum mechanics.

If we omit the assumption of the completeness of H, we obtain the family of quadratic spaces which possesses as a proper subfamily the class of all Hilbert spaces. Different kinds of subspaces have been used for modeling quantum structures and therefore it is very important to know the completeness criteria of quadratic spaces.

One of the outstanding problems of the theory of orthomodular lattices is a characterization of orthomodular lattices to be isomorphic to $L(H)$ for some Hilbert space H. Many specialists have thought that such properties as atomicity, exchange axiom, infinite-dimensionality, and irreducibility of a complete orthomodular lattice are characteristics only of $\mathcal{L}(H)$. Therefore, a result of Keller (1980) was a great surprise for quantum logicians when he presented an OML with all the above properties which cannot be embedded into $\mathcal{L}(H)$ for any H.

The crucial results show that important classes of OMLs are geometries which can be realized by a vector space E over a division ring K equipped with a Hermitian form and with a system of subspaces of E (Piron, 1976; Varadarajan, 1968; Maeda and Maeda, 1970). These results initiated a deep study of connections between orthomodular lattices and quadratic spaces.

Let E be a quadratic space, i.e., E is a vector space over a division ring K with a Hermitian form $(\cdot, \cdot)$. For any subset $M \subseteq E$, we put $M^\perp = \{x \in E: (x, y) = 0 \text{ for any } y \in M\}$. Let $\mathcal{L}(E)$ denote the family of all orthogonally closed subspaces of E, i.e.,

$$\mathcal{L}(E) = \{M \subseteq E: M^{\perp\perp} = M\} \tag{1.3}$$

and let $\mathcal{E}(E)$ denote the set of all splitting subspaces of E, i.e.,

$$\mathcal{E}(E) = \{M \subseteq E: M^\perp + M = E\} \tag{1.4}$$

Then

$$\mathcal{E}(E) \subseteq \mathcal{L}(E)$$

(see below), and E is said to be orthomodular iff $\mathcal{L}(E) \subseteq \mathcal{E}(E)$.

The Amemiya–Araki–Piron result (Amemiya and Araki, 1966/67; Piron, 1976) says that a real or complex inner product space E is complete iff $\mathcal{L}(E)$ is an orthomodular lattice, or equivalently, iff E is an orthomodular space. Keller's (1980) result is the first example of a non-Hermitian orthomodular quadratic space over a non-Archimedian ordered ring. Important contributions are also Morash's (1973) notion of an angle-bisecting system and ones made by Gross and his school (see, e.g., Gross, 1990).

Recently Solèr (1995) presented a very nice and surprising result that any infinite-dimensional orthomodular space containing a sequence of orthonormal vectors[2] is either a real, complex, or quaternionic Hilbert space.

In the present paper, we show that the test space $(E\setminus\{0\}, \mathcal{M}(E))$, where $\mathcal{M}(E)$ consists of all maximal orthogonal systems of E, is algebraic iff E is Dacey, or, equivalently, iff E is orthomodular. In addition, we present another orthomodularity criterion for quadratic spaces, and we show that it can force that E is a real, complex, or quaternionic Hilbert space.

2. QUADRATIC SPACES

Let K be a division ring with char$K \neq 2$ and with an involution $*$: $K \to K$ such that $(\alpha + \beta)^* = \alpha^* + \beta^*$, $(\alpha\beta)^{**} = \beta^*\alpha^*$, $\alpha^{**} = \alpha$ for all α, $\beta \in K$. Let E be a (left) vector space over K equipped with a Hermitian form $(\cdot, \cdot)$: $E \times E \to K$, i.e., $(\cdot, \cdot)$ satisfies, for all $x, y, z \in E$ and all α, $\beta \in K$, $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$, $(x, \alpha y + \beta z) = (x, y)\alpha^* + (x, z)\beta^*$, $(x, y) = (y, x)^*$. The triplet $(E, K, (\cdot, \cdot))$ is said to be a *quadratic space* (or an *inner product space*) if $(x, y) = 0$ for any $y \in E$ implies $x = 0$, and unless confusion threatens, we usually refer to E rather than to $(E, K, (\cdot, \cdot))$.

A *length* of a vector $x \in E$ we call the expression (x, x). A nonzero vector $x \in E$ such that $(x, x) = 0$ is said to be isotropic. E is said to be *anisotropic* if it has no isotropic vectors. For two subspaces M and N of E we write $M \perp N$ iff $(x, y) = 0$ for all $x \in M$ and for all $y \in N$; it is clear that $M \perp N$ iff $N \perp M$. Similarly we write $x \perp y$ if $(x, y) = 0$.

Let $\mathcal{L}(E)$ and $\mathcal{E}(L)$ be defined by (1.3) and (1.4). In general, $\{0\}$ and E belong to $\mathcal{E}(E)$, and the system $\mathcal{L}(E)$ is an atomistic, complete, orthomodular lattice with covering property (Maeda and Maeda, 1970, Theorem 34.2).[3] We recall that if $\{M_i\}$ is a system of subspaces from $\mathcal{L}(E)$, then

$$\bigvee_i M_i = (\bigcup_i M_i)^{\perp\perp} = (\mathrm{sp}(\bigcup_i M_i))^{\perp\perp}, \qquad \bigwedge_i M_i = \bigcap_i M_i \tag{2.1}$$

$$(\bigvee_i M_i)^{\perp} = \bigcap_i M_i^{\perp} \tag{2.2}$$

where sp denotes the span, and if $M \in \mathcal{L}(E)$ and $\dim N < \infty$, then

$$M \vee N = M + N$$

[2] It is sufficient to suppose that there is an infinite sequence of mutually orthogonal vectors of the same length.

[3] A nonzero element a of a poset L is said to be an *atom* of L if $b \leq a$ for some $b \in L$ implies either $b = 0$ or $b = a$. L is said to be *atomistic* if any nonzero element $a \in L$ is the join of atoms contained in a. We say that $b \in L$ covers $a \in L$, and we write $a \lessdot b$ if $a < b$ and moreover $a < c < b$ is not satisfied by any c. L has the *covering property* if the statement p is an atom and $a \wedge p = 0$ implies $a \lessdot a \vee p$.

We note that always

$$\mathscr{E}(E) \subseteq \mathscr{L}(E) \tag{2.3}$$

Indeed, let $M + M^{\perp} = E$ and since $M \subseteq M^{\perp\perp}$, it suffices to prove that $M^{\perp\perp} \subseteq M$. We note first that $\{0\} = E^{\perp} = (M + M^{\perp})^{\perp} = M^{\perp} \cap M^{\perp\perp}$. Let $x \in M^{\perp\perp}$. Then there exist $x_1 \in M$ and $x_2 \in M^{\perp}$ such that $x = x_1 + x_2$. Hence $x_2 = x - x_1 \in M^{\perp\perp}$ and therefore $x_2 = 0$ and, consequently, $x \in M$.

The family $\mathscr{E}(E)$ is an OMP for which $M \perp N$ implies $M \vee N \in \mathscr{E}(E)$, and in this case $M \vee N = M + N$. In general, $\mathscr{E}(E)$ can be neither a σ-OMP even if E is a real or complex inner product space (Dvurečenskij, 1993, Theorem 4.1.6) nor a lattice. For example, Morales and Garcia-Mazaria (n.d.) considered $(\mathrm{R}^4, \mathrm{R}, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is a Hermitian form for the Minkowski space-time:

$$\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle := x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4$$

Then $\mathscr{E}(\mathrm{R}^4)$ is not lattice, and $x = (1, -1, 0, 0,)$ is a nonzero isotropic vector in R^4.

A quadratic space E is said to be *orthomodular* iff $\mathscr{L}(E) \subseteq \mathscr{E}(E)$. If x is a nonzero vector in E, by $\mathrm{sp}(x)$ we denote a one-dimensional subspace of E spanned over x. It is simple to show (Holland, 1995) that $\mathrm{sp}(x) \in \mathscr{L}(E)$ for any $x \in E \setminus \{0\}$, and an orthomodular space is anisotropic. In addition, a one-dimensional subspace $\mathrm{sp}(x)$ belongs to $\mathscr{E}(E)$ iff x is *anisotropic*, i.e., if x is not isotropic. If E is finite-dimensional, then the anisotropy of E implies the orthomodularity of E.

If E is anisotropic, the $\vee_i M_i \in \mathscr{E}(E)$ for a system of splitting subspaces $\{M_i\}$ of E iff $(\cup_i M_i)^{\perp\perp} \in \mathscr{E}(E)$, and then $\vee_i M_i = (\cup_i M_i)^{\perp\perp}$. In an analogous way $\wedge_i M_i \in \mathscr{E}(E)$ iff $\cap_i M_i \in \mathscr{E}(E)$, and in this case $\wedge_i M_i = \cap_i M_i$. In other words, the joins and meet taken in $\mathscr{L}(E)$ and in $\mathscr{E}(E)$ coincide when they exist.

Let M be a subspace of an anisotropic E. We say that a system of nonzero mutually orthogonal vectors $\{x_i\}$ from M is a *maximal orthogonal system* (MOS in short) in M if, $x \in M$, $x \perp x_i$ for any i, imply $x = 0$. It is worth to recall that if E is a real or complex inner product space, then all MOSs have the same cardinality. In general, this is not the case (Gross, 1990, Theorem 5.1).

It is easy to verify that if $\{x_i\}$ is an MOS in a splitting subspace M of an anisotropic E, then

$$\bigvee_i \mathrm{sp}(x_i) = \{x_i\}^{\perp\perp} = M \tag{2.4}$$

We say that a quadratic space E is *Dacey* if, for any MOS $\{x_i\} \cup \{y_j\}$ in E with $\{x_i\} \cap \{y_j\} = \emptyset$, we have

$$\{x_i\}^{\perp\perp} = \{y_j\}^{\perp} \tag{2.5}$$

Let E be a quadratic space and define $E_0 := E\setminus\{0\}$ and let $\mathcal{T}(E_0)$ be the system of all MOSs in E. Then the pair $(E_0, \mathcal{T}(E_0))$ is a test space, and denote by $\mathcal{E}(E_0)$ the system of all events in E_0.

Theorem 2.1. Let E be an anisotropic quadratic space. Then the test space $(E_0, \mathcal{T}(E_0))$ is algebraic if and only if E is Dacey.

Proof. Suppose that $(E_0, \mathcal{T}(E_0))$ is an algebraic test space and choose an MOS $\{x_i\} \cup \{y_j\}$ in E with $\{x_i\} \cap \{y_j\} = \emptyset$. We have to prove that if $A = \{x_i\}$ and $B = \{y_j\}$, then $A^\perp = B^{\perp\perp}$. It is evident that $A \subseteq B^\perp$, so that $B^{\perp\perp} \subseteq A^\perp$. Let now x be any nonzero vector in $A^\perp$; we show now that $x \in B^{\perp\perp}$. Complete $\{x_i\} \cap \{x\}$ by $C_1 = \{z_k\}$ to $\{x_i\} \cup \{x\} \cup \{z_k\}$ to be a MOS in E.[4] The algebraicity of $(E_0, \mathcal{T}(E_0))$ entails that $\pi(\{x\} \cup \{z_k\}) = \pi(\{y_j\})$, where π is defined by (1.1).

Any event in $\mathcal{E}(E_0)$ is either the empty set or an orthogonal system of nonzero vectors in E, and conversely. It is clear that if M, N are two events in $\mathcal{E}(E_0)$, then $M \perp N$ as events iff $(x, y) = 0$ for all $x \in M$, $y \in N$. We define $M^{\perp \mathcal{T}} := \{y \in E_0\colon (x, y) = 0 \text{ for all } x \in M\}$.

Denote $C = \{x\} \cup C_1$. We assert that $\pi(C) = \pi(B)$ iff $C^{\perp\mathcal{T}} = B^{\perp\mathcal{T}}$. Indeed, let $\pi(C) = \pi(B)$ and let $\{z\} \perp C$. By algebraicity $\{z\} \perp B$, so that $z \in B^{\perp\mathcal{T}}$ and, by symmetry, $C^{\perp\mathcal{T}} = B^{\perp\mathcal{T}}$. Conversely, let $C^{\perp\mathcal{T}} = B^{\perp\mathcal{T}}$. If C_1 is a local complement of C, then $C_1 \subseteq C^{\perp\mathcal{T}}$, which yields $C_1 \subseteq B^{\perp\mathcal{T}}$, hence $C_1 \perp B$, therefore $\pi(C)' \le \pi(B)'$. If C' is a local complement of C, then in $\Pi(\mathcal{E}(E_0))$, we have $\pi(C') = \pi(C)'$. Hence, $\pi(B) \le \pi(C)$ and, by symmetry, $\pi(C) \le \pi(B)$.

Therefore, the equality $\pi(C) = \pi(B)$ implies $C^{\perp\mathcal{T}} = B^{\perp\mathcal{T}}$ and $C^\perp = B^\perp$, which is equivalent to $C^{\perp\perp} = B^{\perp\perp}$, so that $x \in B^{\perp\perp}$, which proves $A^\perp \subseteq B^{\perp\perp}$, and E is Dacey.

Conversely, let E be a Dacey quadratic space. We claim the algebraicity of $(E_0, \mathcal{T}(E_0))$. So let $A = \{x_i\}$ and $B = \{y_j\}$ be two events in $\mathcal{E}(E_0)$ which share a common local complement $C = \{z_k\}$, and let $C_1 = \{u_s\}$ be any local complement of B. We can show that C_1 is a local complement of A, too, i.e., $\{x_i\} \cup \{u_s\}$ is an MOS in E.

Since E is Dacey, $A^{\perp\perp} = C^\perp$, $B^{\perp\perp} = C^\perp = C_1^\perp$, and $A^{\perp\perp} = B^{\perp\perp}$. Let $x \perp x_i, u_s$ for all i and s. Then $x \in A^\perp = C^{\perp\perp} = C_1^{\perp\perp}$ and $x \in C_1^\perp$, which gives $x = 0$, so that $\{x_i\} \cup \{u_s\}$ is an MOS in E. ∎

3. ORTHOMODULARITY CRITERIA

In the present section, we give some criteria implying the orthomodularity of an anisotropic quadratic space E. Dvurečenskij (1993, Theorem 5.4.2)

[4]Throughout the paper always when we complete an orthogonal system $\{u_i\}$ from E_0 by an orthogonal system $\{v_j\}$ from E_0 such that $\{u_i\} \cup \{v_j\}$ is an MOS in E, we suppose that $\{u_i\} \cap \{v_j\} = \emptyset$.

shows that E is orthomodular iff $\{x_i\}^{\perp\perp} \in \mathscr{E}(E)$ for any system of mutually orthogonal vectors $\{x_i\}$ in E. In particular (Dvurečenskij, 1993, Theorem 4.1.6), if E is a real or complex inner product space, then E is orthomodular (i.e., E is a Hilbert space) iff $\{x_i\}^{\perp\perp} \in \mathscr{E}(E)$ for any sequence $\{x_i\}$ of orthonormal vectors in E.

We now introduce the following systems of subspaces of E:

(1) $\mathscr{D}(E)$ is the set of all *Foulis–Randall subspaces of* E, i.e., of all subspaces M for which there exists an orthogonal system of nonzero vectors (OS, for short) $\{u_i\}$ such that $M = \{u_i\}^{\perp\perp}$, which is a complete orthoposet. Any M of $\mathscr{D}(E)$ possesses at least one local complement M', i.e., such an element $M' \in \mathscr{D}(E)$ for which $M' \perp M$ and $M \vee M' = E$.

(2) $\mathscr{R}(E)$ is the set of all subspaces M of E such that $M = \{u_i\}^{\perp\perp}$ for all MOSs $\{u_i\}$ of M, which is a poset.

(3) $\mathscr{V}(E)$ is the set of all subspaces M of E such that $M = \{u_i\}^{\perp\perp}$ and $M^\perp = \{v_j\}^{\perp\perp}$ for all MOSs $\{u_i\}$ and $\{v_j\}$ of M and $M^\perp$, respectively, which is an orthocomplemented poset.

It is easy to see that

$$\mathscr{E}(E) \subseteq \mathscr{V}(E) \subseteq \mathscr{R}(E) \subseteq \mathscr{D}(E) \subseteq \mathscr{L}(E) \tag{3.1}$$

Let $\mathcal{M}$ be a system of subspaces of a quadratic space E. We say that $\mathcal{M}$ has the *orthomodular property* iff $A, B \in \mathcal{M}$ with $A \subseteq B$ implies $B = A \vee (B \cap A^\perp)$, where the join $\vee$ is defined by the left-hand side of (2.1).

Lemma 3.1. Let E be an arbitrary quadratic space. Then E is orthomodular if and only if $\mathscr{L}(E)$ has the orthomodular property.

Proof. If E is an orthomodular quadratic space, then $\mathscr{L}(E) = \mathscr{E}(E)$ and $\mathscr{E}(E)$ has always the orthomodular property.

Conversely, let $\mathscr{L}(E)$ have the orthomodular property and let $M \in \mathscr{L}(E)$ be given. We claim that $M \in \mathscr{E}(E)$. Let $x \in E$. If $x \in M \cup M^\perp$, then easily $x \in M + M^\perp$. Suppose now $x \notin M \cup M^\perp$. Then $\mathrm{sp}(x) \in \mathscr{L}(E)$, and the orthomodularity of $\mathscr{L}(E)$ yields $M \vee \mathrm{sp}(x) = M \vee ((M \vee \mathrm{sp}(x)) \cap M^\perp)$, which gives $((M \vee \mathrm{sp}(x)) \cap M^\perp \neq \{0\}$. Hence, there is a nonzero vector $z \in M^\perp$ and $z \in M \vee \mathrm{sp}(x) = M + \mathrm{sp}(x)$, so that there are $z_1 \in M$ and a nonzero $\alpha \in K$ with $z = z_1 + \alpha x$. Then $x = x_1 + x_2$, where $x_1 := -\alpha^{-1} z_1 \in M$ and $x_2 := \alpha^{-1} x \in M^\perp$. ∎

Theorem 3.2. Let any MOS in an anisotropic quadratic space E be at most countable. Then E is orthomodular if and only if $\mathscr{D}(E)$ has the orthomodular property.

Proof. Suppose that $\mathscr{D}(E)$ has the orthomodular property. We claim that then $\mathscr{L}(E) = \mathscr{D}(E)$. Let $M \in \mathscr{L}(E)$ and let $\{x_i\}$ be an MOS in M. Put $N = \{x_i\}^{\perp\perp}$. Then $N \subseteq M$ and we assert that $N = M$. If not, then there exists a nonzero vector $x \in M \setminus N$. Applying the Gram–Schmidt orthogonalization process to the set $\{x, x_1, x_2, \ldots\}$, we obtain an orthogonal system $\{x, e_1, e_2, \ldots\}$, and $\{x, e_1, e_2, \ldots\}^{\perp\perp} \in \mathcal{D}(E)$. This implies $N \vee \mathrm{sp}(x) = \{x, e_1, e_2, \ldots\}^{\perp\perp} \in \mathcal{D}(E)$. The orthomodularity of $\mathcal{D}(E)$ implies $N \vee \mathrm{sp}(x) = N \vee ((N \vee \mathrm{sp}(x)) \cap N^{\perp})$. Since $x \notin N$, we have $(N \vee \mathrm{sp}(x)) \cap N^{\perp} \neq \{0\}$. Therefore, there is a nonzero vector $z \in (N \vee \mathrm{sp}(x)) \cap N^{\perp}$ which yields $z \in M$, so that $\{x_i\} \cup \{z\}$ is an OS, which contradicts the maximality of $\{x_i\}$ in M. Therefore, our assumption that $N \neq M$ was false, so that $N = M$. This means that $\mathcal{L}(E) = \mathcal{D}(E)$, and by Lemma 3.1, E is orthomodular. ■

Remark 3.3. Let either (i) $N \in \mathcal{D}(S)$ and $x \neq 0$ from an anisotropic inner product space E imply $N \vee \mathrm{sp}(x) \in \mathcal{D}(S)$, or (ii) let in any $M \in \mathcal{L}(E)$ there exist at most countable MOS. Then E is orthomodular if and only if $\mathcal{D}(S)$ has the orthomodular property.

Proof. (i) The proof is same as that of Theorem 3.2 without using the Gram–Schmidt orthogonalization process. For (ii) we repeat the proof of Theorem 3.2. ■

We remark that if E is a real or complex inner product space of any dimension, then E is a Hilbert space iff $\mathcal{D}(E)$ has the orthomodular property (Dvurečenskij and Pulmannov'a, 1994, Theorem 3.1). Therefore, we do not know whether the condition on countability of any MOS in Theorem 3.2 is superfluous in order to be E orthomodular.

Lemma 3.4. An anisotropic quadratic space E is Dacey if and only if any MOS $\{x_k\}$ in E with $\{x_k\} \subseteq \{x\}^{\perp} \cup \{y\}^{\perp}$ for some vectors $x, y \in$ implies $x \perp y$.

Proof. Let E be Dacey and let $\{x_k\}$ with $\{x_k\} \subseteq \{x\}^{\perp} \cup \{y\}^{\perp}$ be given. Define $\{u_i\} := \{x_k\} \cap \{x\}^{\perp}$ and $\{v_j\} := \{x_k\} \setminus \{u_i\}$. Then $\{u_i\} \subseteq \{x\}^{\perp}$ and $\{v_j\} \subseteq \{y\}^{\perp}$, so that $x \in \{u_i\}^{\perp}$ and $y \in \{v_j\}^{\perp}$ and since E is Dacey, we have $x \perp y$.

On the contrary, let $\{x_i\} \cup \{y_j\}$ with $\{x_i\} \cap \{y_j\} = \emptyset$ be an MOS in E. Then $\{x_i\} \subseteq \{y_j\}^{\perp}$, so that $\{x_i\}^{\perp\perp} \subseteq \{y_j\}^{\perp}$. If now $x \in \{x_i\}^{\perp}$ and $y \in \{y_j\}^{\perp}$, then

$$\{x_i\} \cup \{y_j\} \subseteq \{x_i\}^{\perp\perp} \cup \{y_j\}^{\perp\perp} \subseteq \{x\}^{\perp} \cup \{y\}^{\perp}$$

which gives $x \perp y$, and hence $\{x_i\}^{\perp} \perp \{y_j\}^{\perp}$, so that $\{y_j\}^{\perp} \subseteq \{x_i\}^{\perp\perp}$. ■

We recall that if, for an OS $\{x_i\} \cup \{y_j\}$ with $\{x_i\} \cap \{y_j\} = \emptyset$, we have (2.5), then $\{x_i\} \cup \{y_j\}$ is an MOS in E.

Let $M \in \mathcal{D}(E)$; then an element $M' \in \mathcal{D}(E)$ such that $M' \perp M$ and $M \vee M' = E$ is said to be a *local complement* of M in $\mathcal{D}(E)$.

Theorem 3.5. An anisotropic quadratic space E is Dacey if and only if, for any $M \in \mathcal{D}(E)$, $M^\perp$ is a unique local complement of M in $\mathcal{D}(E)$.

Proof. Let $M \in \mathcal{D}(E)$ and let $M = \{u_i\}^{\perp\perp}$ for some OS $\{u_i\}$ in E. Complete $\{u_i\}$ by an OS $\{v_j\}$ such that $\{u_i\} \cup \{y_j\}$ is an MOS in E. Since E is Dacey, $M = \{u_i\}^{\perp\perp} = \{v_j\}^\perp$ and $M^\perp = \{v_j\}^\perp \in \mathcal{D}(E)$. It is clear that $M^\perp$ is a local complement of M. Suppose that $M' \in \mathcal{D}(E)$ is another local complement of M in $\mathcal{D}(E)$ and let $M' = \{w_k\}^{\perp\perp}$. Then $M' \subseteq M^\perp$, $\{w_k\} \cup \{u_j\}$ is an MOS in E. We assert that $M' = M^\perp$. If not, then $M \subseteq M'^\perp$ and $M \neq M'^\perp$, and there exists a nonzero vector $x \in M'^\perp \backslash M$. Hence $x \in \{w_k\}^\perp$ and there exists $y \in \{u_i\}^\perp$ with $y \not\perp x$. Then $\{w_k\} \subseteq \{w_k\}^{\perp\perp} \subseteq \{x\}^\perp$ and $\{u_i\} \subseteq \{y\}^\perp$. According to Lemma 3.4, this contradicts that E is Dacey.

Conversely, let $M \in \mathcal{D}(E)$ and let $M^\perp \in \mathcal{D}(E)$ be a unique local complement of M in $\mathcal{D}(E)$. Let $\{x_i\} \cup \{y_j\}$ with $\{x_i\} \cap \{y_j\} = \emptyset$ be an MOS in E. Define $M = \{x_i\}^{\perp\perp}$ and $M' = \{y_j\}^{\perp\perp}$. Then $M \perp M'$, $M, M' \in \mathcal{D}(E)$. Since $M \vee M' = \{x_i\}^{\perp\perp} \vee \{y_j\}^{\perp\perp} = (\{x_i\} \cup \{y_j\})^{\perp\perp} = E$, we conclude that E is Dacey. ∎

Theorem 3.6. An anisotropic quadratic space E is Dacey if and only if $\mathcal{V}(E) = \mathcal{D}(E)$.

Proof. Let $M \in \mathcal{D}(E)$ and let $\{u_i\}$ be an OS such that by definition $M = \{u_i\}^{\perp\perp}$. Complete $\{u_i\}$ by an OS $\{v_j\}$ such that $\{u_i\} \cup \{v_j\}$ is an MOS in E. Choose an arbitrary MOS $\{z_k\}$ in M. We claim that $\{z_k\} \cup \{v_j\}$ is an MOS in E. If not, complete $\{z_k\} \cup \{v_j\}$ by an OS $\{y_s\}$ such that $\{z_k\} \cup \{v_j\} \cup \{y_s\}$ is an MOS in E. Since E is Dacey, $M^\perp = \{v_j\}^\perp$; consequently, $M = (\{z_k\} \cup \{y_s\})^{\perp\perp}$, which contradicts the maximality of $\{z_k\}$ in M; consequently we have proved that $M = \{z_k\}^{\perp\perp}$ for any MOS $\{z_k\}$ in M, so that $\mathcal{D}(E) \subseteq \mathcal{R}(E)$. Since $M^\perp = \{v_j\}^{\perp\perp} \in \mathcal{D}(E) = \mathcal{R}(E)$, we conclude that $M \in \mathcal{V}(E)$.

Let $\mathcal{V}(E) = \mathcal{D}(E)$ and suppose that $\{u_i\} \cup \{v_j\}$ with $\{u_i\} \cap \{v_j\} = \emptyset$ is an MOS in E. Putting $M = \{u_i\}^{\perp\perp}$, we have $M \in \mathcal{D}(E) = \mathcal{V}(E)$. Now, $\{v_j\} \perp \{u_i\}$ implies $\{v_j\} \subseteq \{u_i\}^\perp = M^\perp$, but $\{v_j\}$ is a maximal in $M^\perp$, as can be easily seen, so that $\{v_j\}^{\perp\perp} = M^\perp = \{u_i\}^\perp$. ∎

Theorem 3.7. Let any MOS in an anisotropic quadratic space E be at most countable. The following statements are equivalent.

(i) E is orthomodular.
(ii) E is Dacey.
(iii) $(E_0, \mathcal{T}(E_0))$ is an algebraic test space.
(iv) $\mathcal{V}(E) = \mathcal{D}(E)$.
(v) For any $M \in \mathcal{D}(E)$, $M^\perp$ is the unique local complement of M in $\mathcal{D}(E)$.
(vi) $\mathcal{R}(E) = \mathcal{D}(E)$.

Proof. The equivalence of (ii), (iii), (iv), and (v) has been established in Theorems 2.1, 3.5, and 3.6.

(i) $\Rightarrow$ (ii). If now E is orthomodular, then $\mathcal{L}(E) = \mathcal{E}(E)$ so that by (3.1) we conclude that $\mathcal{R}(E) = \mathcal{D}(E)$ and Theorem 3.6 implies that E is Dacey.

(ii) $\Rightarrow$ (i). Suppose E is Dacey. First we show that if $M \in \mathcal{D}(E)$ and $x \in E_0 \backslash M$, then $M \vee \mathrm{sp}(x) \in \mathcal{D}(E)$. Choose an MOS $\{u_i\}$ in M; then by (iv), $\{u_i\}^{\perp\perp} = M$. Applying the Gram–Schmidt orthogonalization process to $\{x, u_1, u_2, \ldots\}$, we obtain an OS $\{x, x_1, x_2, \ldots\}$ such that $\{x, u_1, u_2 \ldots\}^{\perp\perp} = \{x, x_1, x_2, \ldots\}^{\perp\perp}$. Hence, $M \vee \mathrm{sp}(x) = \{x, u_1, u_2, \ldots\}^{\perp\perp} \in \mathcal{D}(E)$.

We now show that $\mathcal{D}(E)$ has the orthomodular property (even without any assumption on cardinalities of MOSs). Indeed, let $A, B \in \mathcal{D}(E)$ with $A \subseteq B$ be given. Then $A = \{u_i\}^{\perp\perp}$ for some OS $\{u_i\}$, and complete $\{u_i\}$ by $\{v_j\}$ such that $\{u_i\} \cup \{v_j\}$ is an MOS in B. Then

$$B \supseteq A \vee (B \cap A^{\perp}) = \{u_i\}^{\perp\perp} \vee (B \cap A^{\perp})$$
$$\supseteq \{u_i\}^{\perp\perp} \vee \{v_j\}^{\perp\perp} = (\{u_i\} \cup \{v_j\})^{\perp\perp} = B$$

Finally, let $M \in \mathcal{L}(E)$ and choose an MOS $\{x_k\}$ in M. Similarly as in the proof of Theorem 3.2, we can prove that $M = \{x_k\}^{\perp\perp}$. Consequently, $\mathcal{L}(E) = \mathcal{D}(E)$ and E is orthomodular.

(i) $\Rightarrow$ (vi). This is evident.

(vi) $\Rightarrow$ (iv). Let $M \in \mathcal{D}(E)$ and choose an MOS $\{u_i\}$ in M. Then $M = \{u_i\}^{\perp\perp} \in \mathcal{R}(E)$. Complete $\{u_i\}$ by an OS $\{v_j\}$ such that $\{u_i\} \cup \{v_j\}$ is an MOS in E. Then $M' = \{v_j\}^{\perp\perp} \subseteq M^{\perp}$. We assert that $M^{\perp} = M'$. Indeed, let x be any nonzero vector in $M^{\perp}$. Applying the Gram–Schmidt orthogonalization process to $\{x, v_1, v_2, \ldots\}$, we obtain an OS $\{x, x_1, x_2, \ldots\}$ such that $\{x, v_1, v_2, \ldots\}^{\perp\perp} = \{x, x_1, x_2, \ldots\}^{\perp\perp}$. Then $M' \vee \mathrm{sp}(x) \in \mathcal{D}(E)$. It is evident that $\{v_i\}$ is an MOS in M' as well as in $M' \vee \mathrm{sp}(x)$. Since $M' \vee \mathrm{sp}(x) \in \mathcal{R}(E)$, we conclude that $M \vee \mathrm{sp}(x) = \{v_i\}^{\perp\perp} = M'$, which implies $x \in M'$ and $M' = M^{\perp}$. ■

An anisotropic quadratic space E is *half-normal* if there is a sequence $\{e_i\}_{i=1}^{\infty}$ of mutually orthogonal vectors such that $(e_i, e_i) = 1$ for any i ($\{e_i\}_i$ is called an orthonormal sequence).

Theorem 3.8. Let E be an infinite-dimensional half-normal anisotropic quadratic space such that any MOS in E is at most countable. The following statements are equivalent.

(i) E is orthomodular.

(ii) E is Dacey.

(iii) $(E_0, \mathcal{T}(E_0))$ is an algebraic test space.

(iv) $\mathcal{V}(E) = \mathcal{D}(E)$.

(v) For any $M \in \mathcal{D}(E)$, $M^{\perp}$ is the unique local complement of M in $\mathcal{D}(E)$.

(vi) $\mathcal{R}(E) = \mathcal{D}(E)$.

(vii) $\{u_i\}^{\perp\perp} \in \mathcal{E}(E)$ for any oS $\{u_i\}$ in E.

(viii) E is a real, complex, or quaternionic separable Hilbert space, dim $E = \aleph_0$.

Proof. The equivalence of (i)–(vi) and (vii) follows from Theorem 3.7 and Dvurečenskij (1993), Theorem 5.4.2. Applying Solèr's theorem to (i), we conclude that E is a real, complex, or quaternionic Hilbert space, dim $E = \aleph_0$. ■

Theorem 3.9. Let E be an anisotropic half-normal quadratic space, dim $E = \aleph_0$, and let all MOSs in $M = \{e_i\}^{\perp\perp}$ have the same cardinality, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal sequence. The following statements are equivalent.

(i) E is orthomodular.

(ii) $\mathcal{D}(E)$ has the orthomodular property.

(iii) E is Dacey.

(iv) $(E_0, \mathcal{T}(E_0))$ is an algebraic test space.

(v) $\mathcal{V}(E) = \mathcal{D}(E)$.

(vi) For any $M \in \mathcal{D}(E)$, $M^{\perp}$ is the unique local complement of M in $\mathcal{D}(E)$.

(vii) $\mathcal{R}(E) = \mathcal{D}(E)$.

(viii) $\{u_i\}^{\perp\perp} \in \mathcal{E}(E)$ for any OS $\{u_i\}$ in E.

(ix) E is a real, complex, or quaternionic Hilbert space.

Proof. Statement (i) implies all the other ones, and (ix) yields (i)–(viii).

(ii) $\Rightarrow$ (i). (a) let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal sequence. Put $M = \{e_i\}^{\perp\perp}$. Then $M \in \mathcal{D}(E) \subseteq \mathcal{L}(E)$.

Define

$$\mathcal{L}(0, M) = \{N \in \mathcal{L}(S): N \subseteq M, N^{\perp_M \perp_M} = N\}$$

$$\mathcal{L}(M) = \{N \subseteq M: N^{\perp_M \perp_M} = N\}$$

$$\mathcal{D}(0, M) = \{N \in \mathcal{D}(S): N \subseteq M\}$$

$$\mathcal{D}(M) = \{N \subseteq M: \{u_i\}^{\perp_M \perp_M} = N \text{ for some ONS } \{u_i\}\}$$

where $N^{\perp_M} = \{x \in M: (x, y) = 0, \forall y \in N\} = N^{\perp} \cap M$. Then

$$\mathcal{L}(0, M) = \mathcal{L}(M) \tag{3.2}$$

$$\mathcal{D}(0, M) = \mathcal{D}(M) \tag{3.3}$$

Indeed, it is evident that $\mathcal{L}(0, M) \subseteq \mathcal{L}(M)$. Conversely, let $N \in \mathcal{L}(M)$; we assert that $N^{\perp\perp} = N$. Calculate

$$N = N \cap M \subseteq N^{\perp\perp} \cap M = (N^{\perp})^{\perp} \cap M = (N^{\perp})^{\perp_M} \subseteq (N^{\perp_M})^{\perp_M} = N$$

so that $N = N^{\perp\perp}$. Here we used the fact that if $A \subseteq M$, then $A^{\perp} \supseteq A^{\perp_M}$.

Let now $N \in \mathcal{D}(0, M)$. Then $N = \{u_i\}^{\perp\perp} \subseteq M$ for some orthonormal system $\{u_i\}$ in N. From (3.2) we have $N^{\perp_M \perp_M} = N^{\perp\perp} = N$. Calculate

$$N = \{u_i\}^{\perp\perp} = \{u_i\}^{\perp\perp} \cap M = (\{u_i\}^{\perp})^{\perp_M} \subseteq \{u_i\}^{\perp_M \perp_M} \subseteq N$$

Hence, $N = \{u_i\}^{\perp_M \perp_M}$ and $N \in \mathcal{D}(M)$.

Conversely, let $N \in \mathcal{D}(M)$; then $N = \{u_i\}^{\perp_M \perp_M} \subseteq M$. By the assumptions, $\{u_i\}$ is countable. Put $N_0 = \{u_i\}^{\perp\perp}$; then $N_0 \subseteq N$. We assert that $N_0 = N$. Take a nonzero vector $x \in N$. Applying the Gram–Schmidt orthogonalization process to $\{x, u_1, u_2, \ldots\}$, we obtain an OS $\{x, v_1, v_2, \ldots\}$, and similarly as in the proof of Theorem 3.2, we obtain that $x \in N_0$.

Applying Theorem 3.2 to the anisotropic quadratic space M, we have from (3.2) and (3.3) that M is an orthomodular half-normal space, dim $M = \aleph_0$, containing $\{e_i\}_{i=1}^{\infty}$. Applying the Solèr theorem to M, we see that M is a real, complex, or quaternionic Hilbert space. In particular, the division ring K of E (which is the same as that of M) is a real, complex, or quaternionic one.

(b) Let now $\{x_i\}$ be any sequence of orthonormal vectors in E. Put $M_1 = \{x_i\}^{\perp\perp}$; then $M_1 \in \mathcal{L}(E)$ and $M_1 \in \mathcal{D}(E)$. It is evident that (3.2) and (3.3) hold also for $M = M_1$, so that $\mathcal{L}(M_1)$ has the orthomodular property, which in view of the theorem of Amemiya–Araki–Piron (holding also for the quaternionic case), implies that M_1 is complete and hence $M_1 \in \mathcal{E}(E)$. By Dvurečenskij (1993), Theorem 4.1.6, this implies that E is complete.

(vii) $\Rightarrow$ (ix). Put as in (a), $M = \{e_i\}^{\perp\perp}$ and define

$$\mathcal{R}(0, M) = \{N \subseteq \mathcal{R}(E): N \subseteq M, N = \{u_i\}^{\perp_M \perp_M} = N$$
$$\text{for any MOS } \{u_i\} \text{ in } N\}$$

$$\mathcal{R}(M) = \{N \subseteq M: N = \{u_i\}^{\perp_M \perp_M} \text{ for any MOS } \{u_i\} \text{ in } N\}$$

In similar manner as that in (a), we can prove that $\mathcal{R}(0, M) = \mathcal{R}(M)$. Therefore, $\mathcal{D}(M) = \mathcal{R}(M)$, which, by (vi) of Theorem 3.8, implies that M is a real, complex, or quaternionic Hilbert space, so that E is a real, complex, or quaternionic quadratic space.

Choose now any mutually orthonormal sequence $\{x_i\}$ in E and define $M_1 = \{x_i\}^{\perp\perp}$. Repeating the same process for M_1, we have $\mathcal{R}(M_1) = \mathcal{D}(M_1)$, so that M_1 is complete, and consequently E is a Hilbert space.

The equivalence of (iii)–(vii) has been established in Theorems 2.1, 3.5, and 3.6 without any assumption on the cardinalities of MOSs. ■

Remark 3.10. Theorems 3.8 and 3.9 are also valid if E is not necessarily half-normal, but if in E there is a sequence of mutually orthogonal nonzero vectors $\{e_i\}$ of the same length. Indeed, we follow ideas of Holland (1995). Let $\lambda = (e_1, e_1)$. Endow K with a new involution $^{\#}$: $K \to K$ by $\alpha^{\#} := \lambda\alpha^*\lambda^{-1}$ and define a new Hermitian form $\langle\cdot, \cdot\rangle$ on E by $\langle\cdot, \cdot\rangle = (\cdot, \cdot)\lambda^{-1}$. Then

$(E, K, \langle \cdot, \cdot \rangle)$ is with respect to $^{\#}$ an anisotropic quadratic space, and orthogonalities defined by $(\cdot, \cdot)$, and $\langle \cdot, \cdot \rangle$ are the same.

Example 3.11. Let Z be the field of all integers and let Z^f be the set of all sequences $(z_1, z_2, \ldots) \in Z^\infty$ such that only finitely many z_i are nonzero. Then Z^f can be assumed as a vector space over Z, and we define a bilinear form $\langle \cdot, \cdot \rangle$: $Z^f \times Z^f \to Z$ via

$$\langle (z_1, z_2, \ldots), (t_1, t_2, \ldots) \rangle = \sum_i z_i t_i$$

Then $(Z^f, Z, \langle \cdot, \cdot \rangle)$ is an infinite-dimensional anisotropic quadratic space. Define $e_i = (0, \ldots, 1, \ldots)$, where 1 is on the ith place of e_i. Then $\{e_i\}_{i=1}^{\infty}$ is an orthonormal sequence, but Z^f is not an orthomodular space (Piziak, 1992, Example 2.3).

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